Machine Learning 2.14: Time

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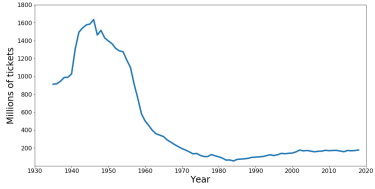
Time

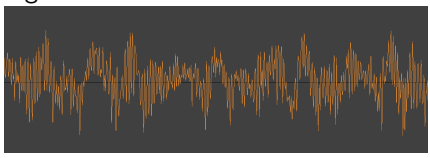
- Time = sequences

UK cinema ticket sales:



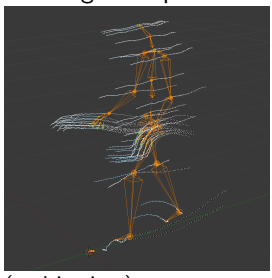
(discrete)

Light saber audio:



(continuous but sampled)

Walking mocap:



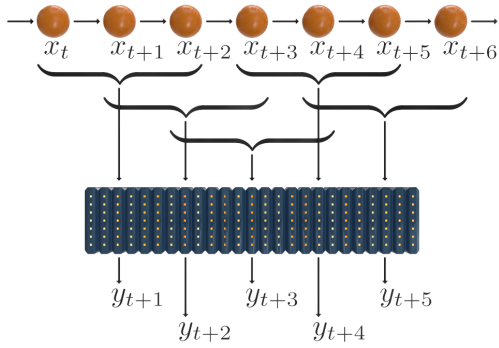
(multivariate)

- Equal sampling most common
- Doesn't have to be time – anything linear

Windows

- Easy approach: Short temporal windows
  (much like convolutional neural networks)
- Can overlap, have stride, arbitrary width etc.

- Lack of temporal coherence
  (noisy)
- Stacking: Second layer to clean up first
  (Mean, voting, ML, Markov random chain. . .)



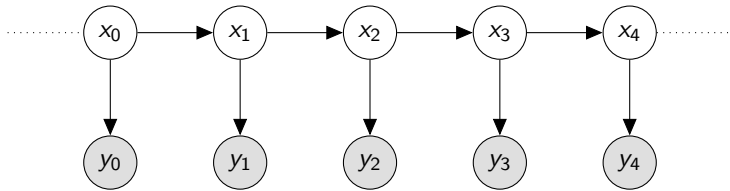
Markov property

- Last discussed: Lecture 12 of Machine Learning 1

- "*Each moment in time depends only on the last*"
- A statement about conditional independence:

$$P(x_0, x_1, x_2, x_3, x_4) = P(x_4|x_3)P(x_3|x_2)P(x_2|x_1)P(x_1|x_0)P(x_0)$$

Hidden Markov model



- x: Unobserved (hidden) Markov chain
- y: Observations, dependent only on simultaneous hidden state

Maximum a posteriori

- Find $\underset{x}{\text{argmax}}\, P(x|y)$
- Belief propagation, again (yay!)

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- Going to walk through HMM MAP specialisation
- Names:
  - Viterbi
  - Dynamic programming

Smoothing

- Many uses of HMM, e.g.
  - Voice recognition
  - Part-of-speech tagging
  - Tracking (video)
  - Genetic information

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    ⋮
- Common use: Smoothing / regularisation
  - Observed: Noisy state
  - Unobserved: True state

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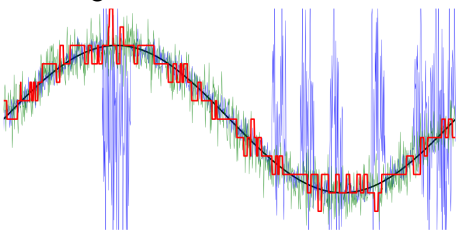
- Noise:
  - Measurement error
  - ML ($\text{state} \approx f(\text{real observation})$)
  - Windows are incorrect...

    ...but work really well!
- Smoothing?



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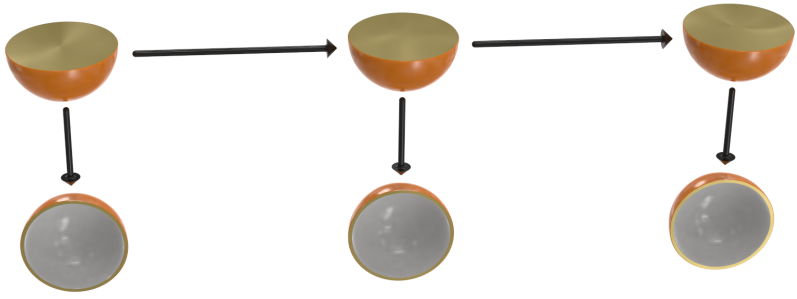
Message passing



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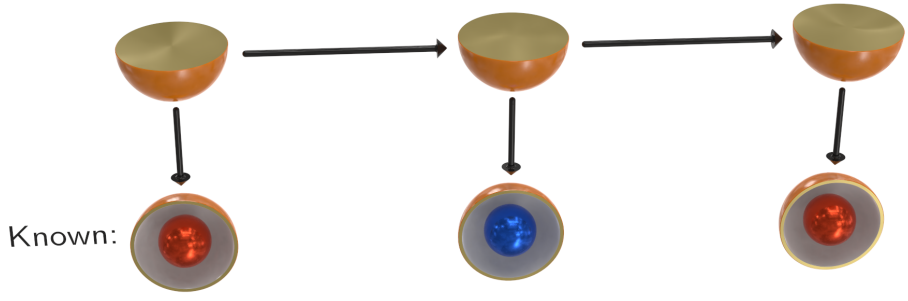

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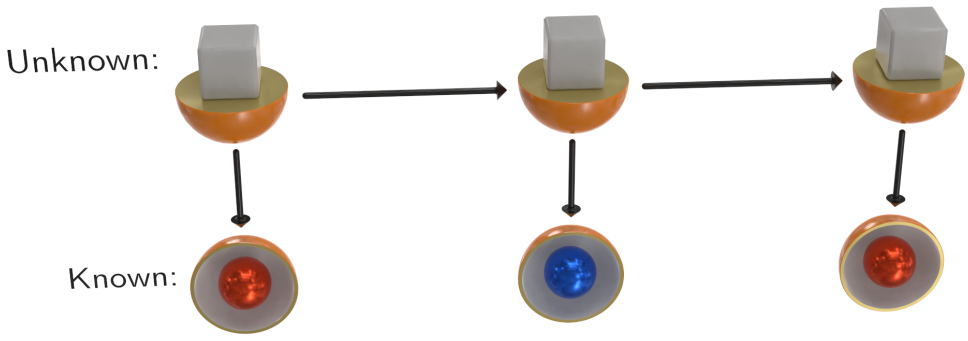

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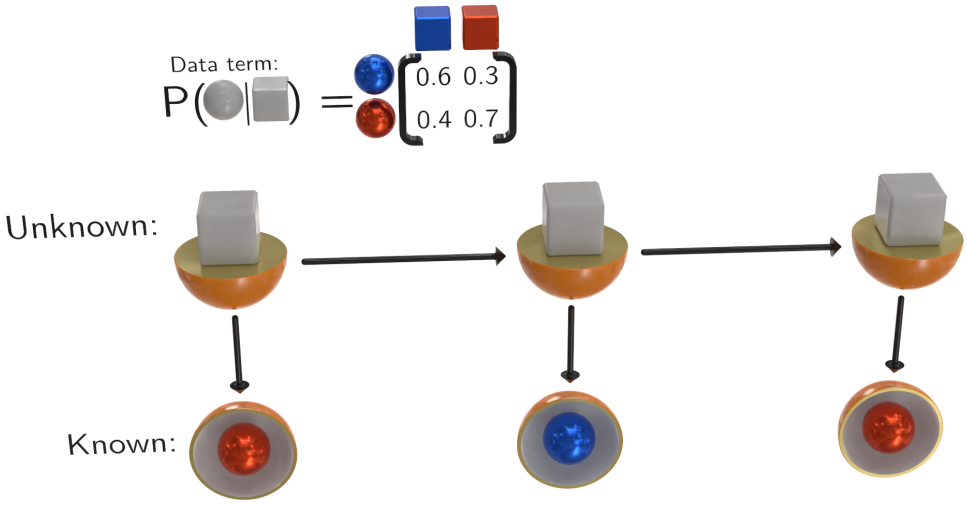

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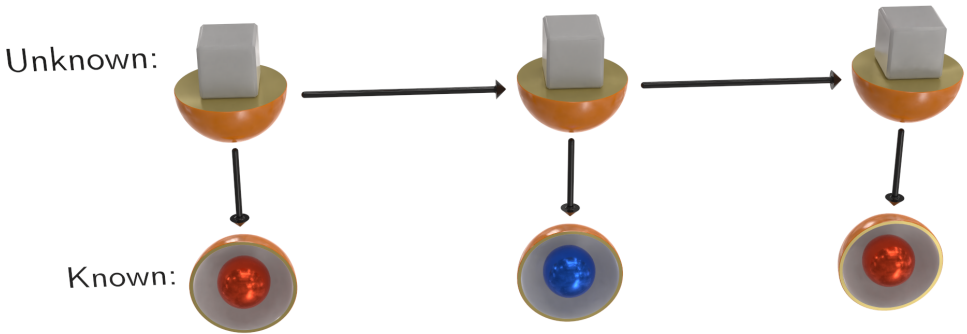

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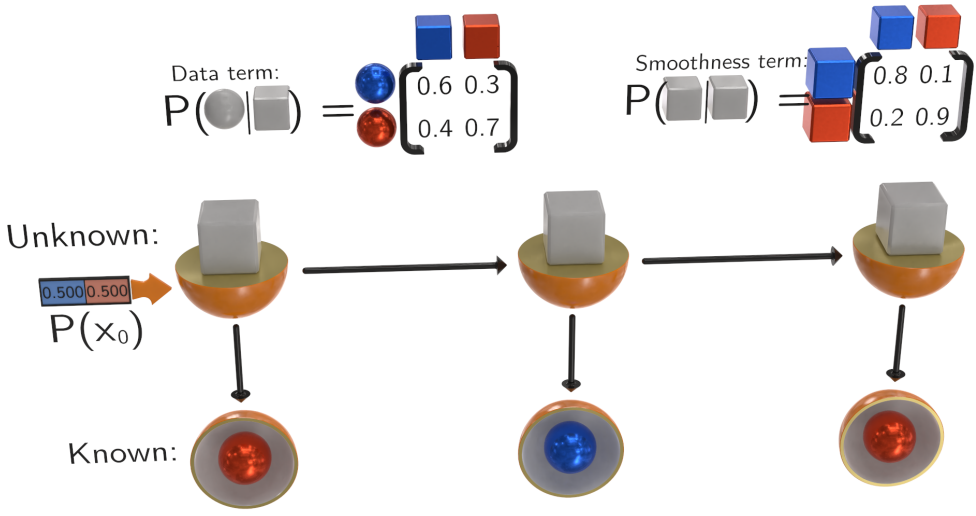

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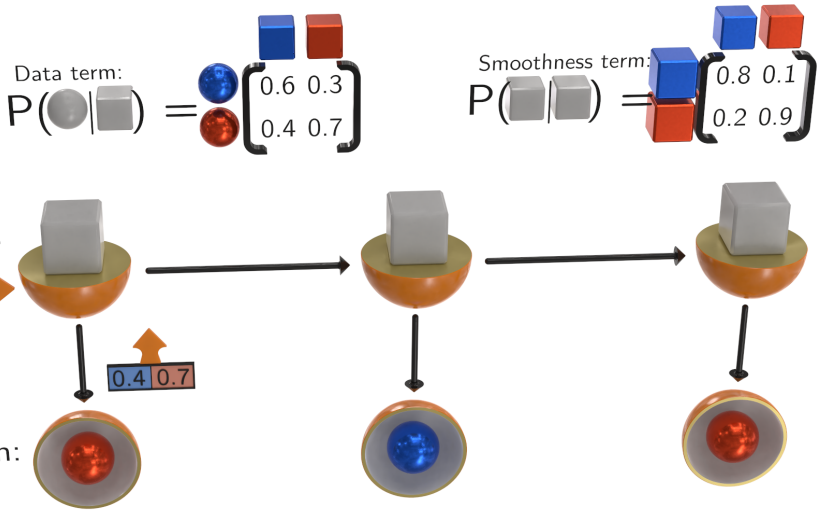

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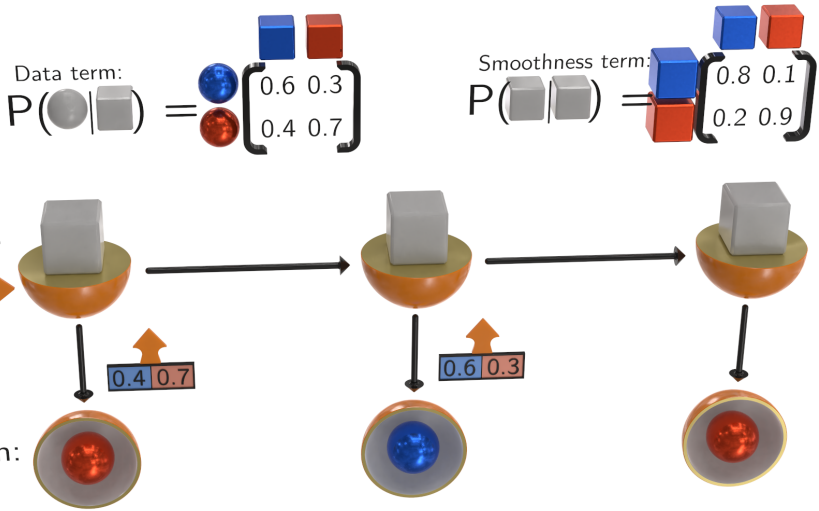

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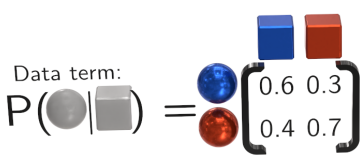

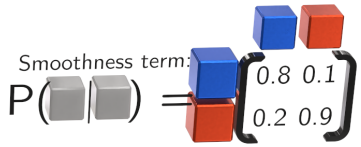

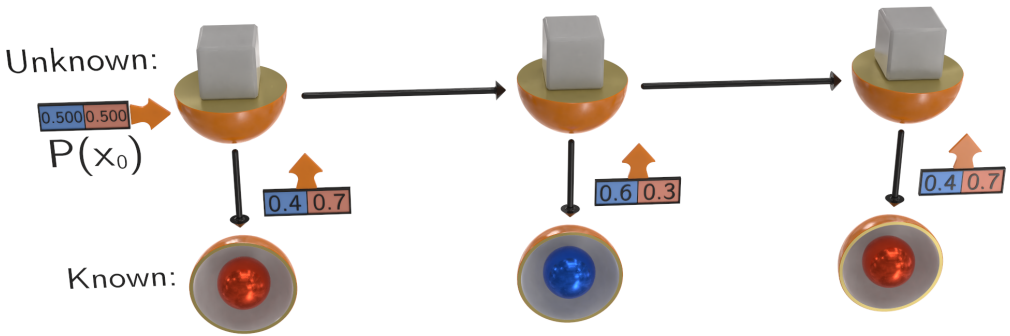

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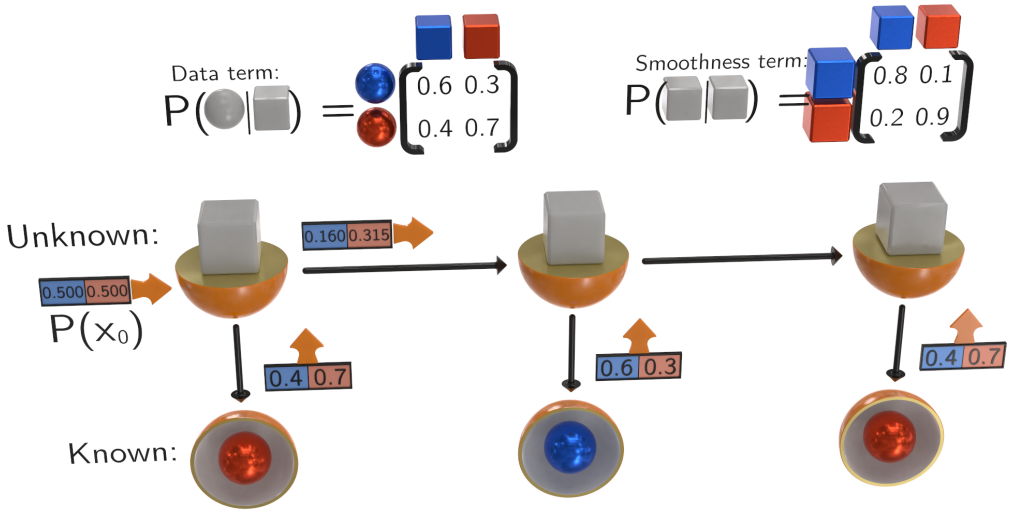

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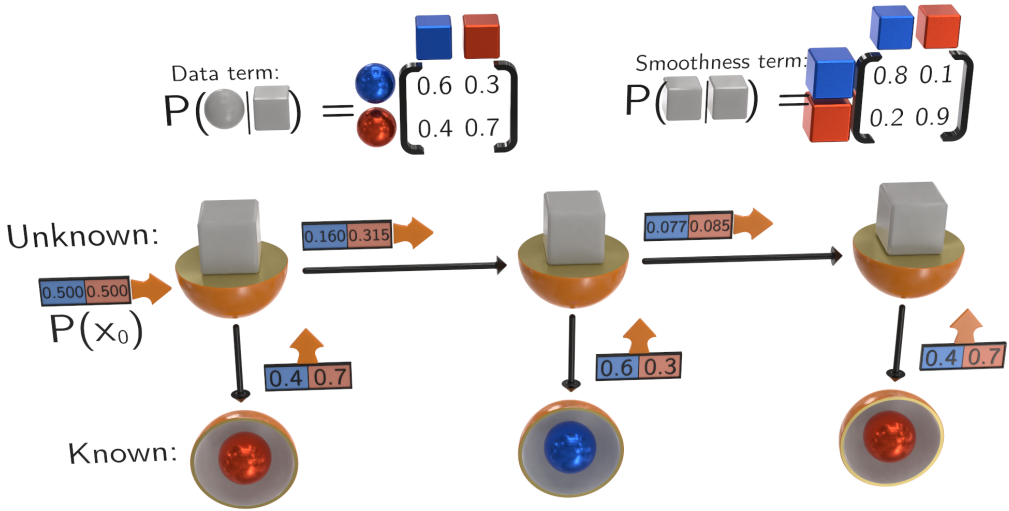

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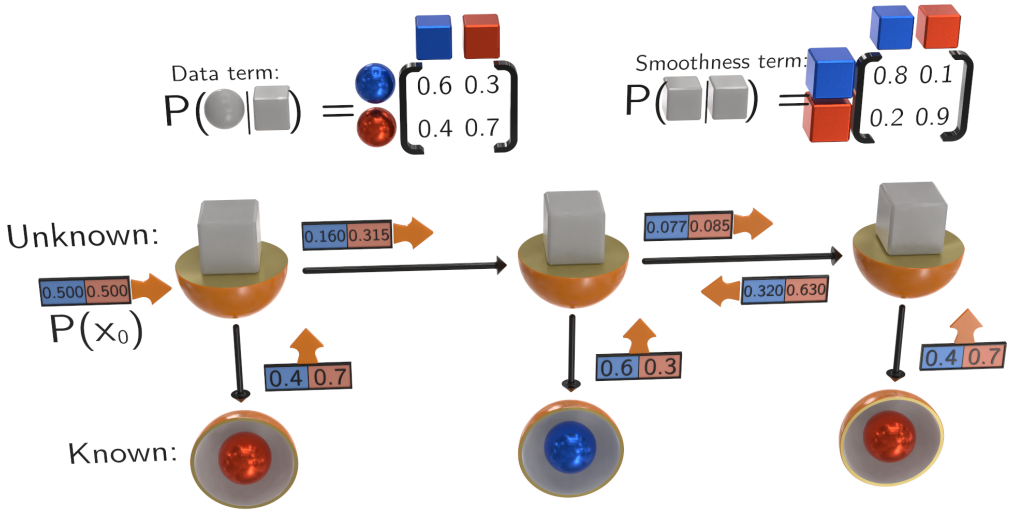

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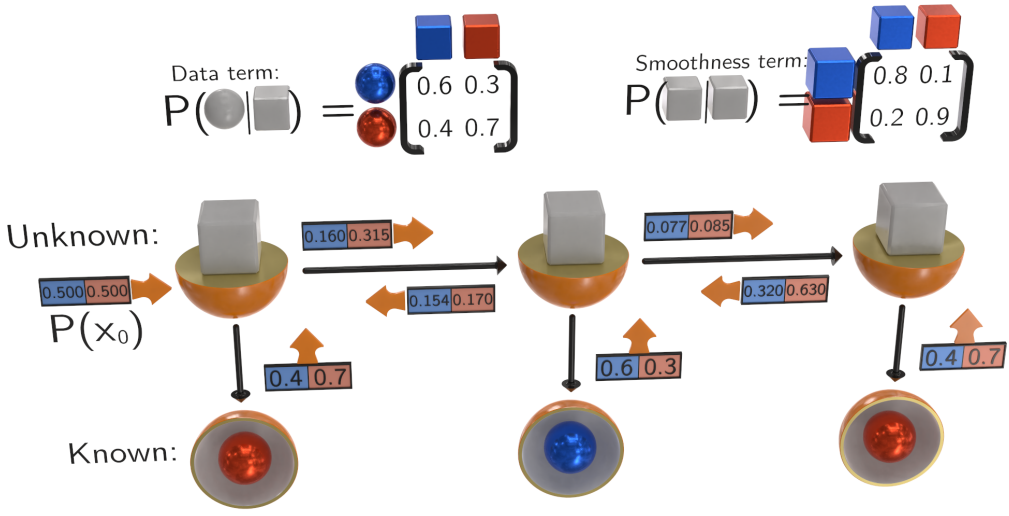

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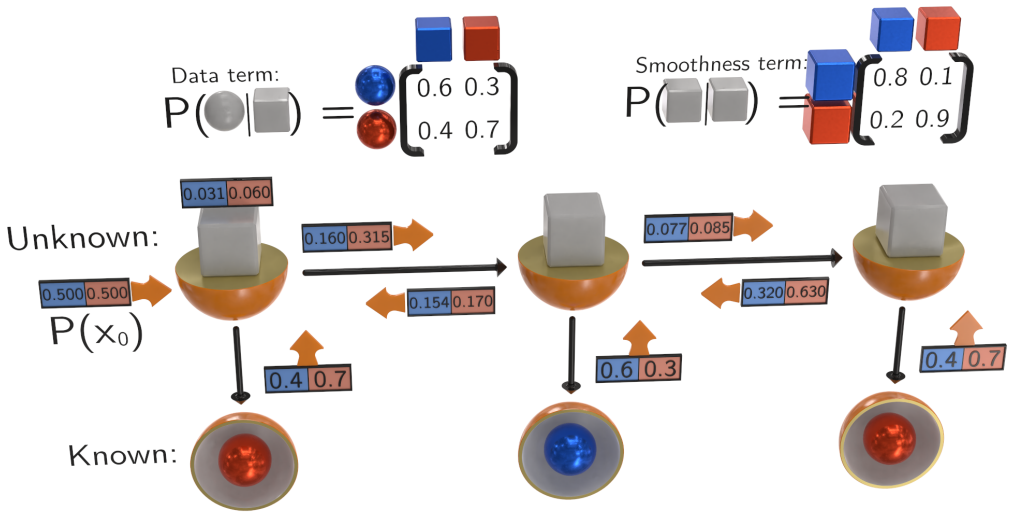

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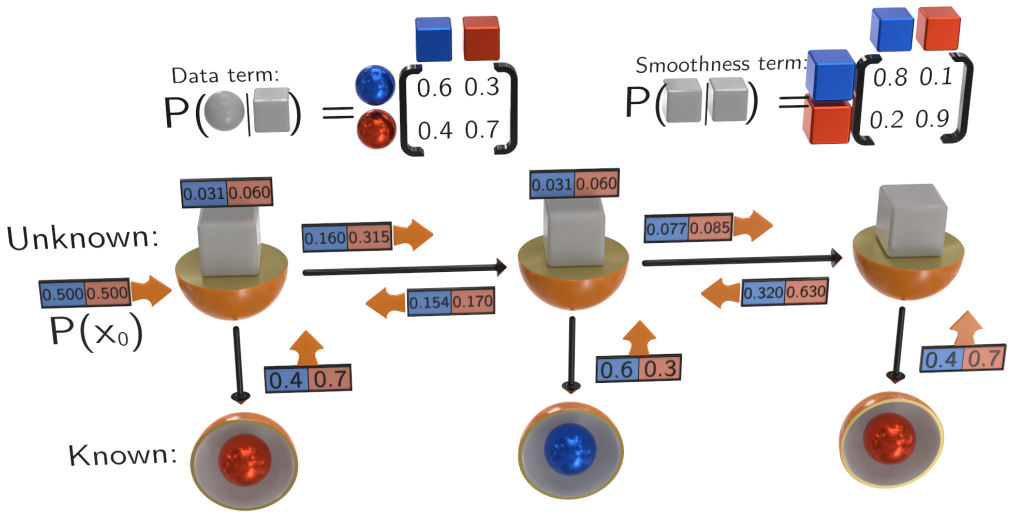

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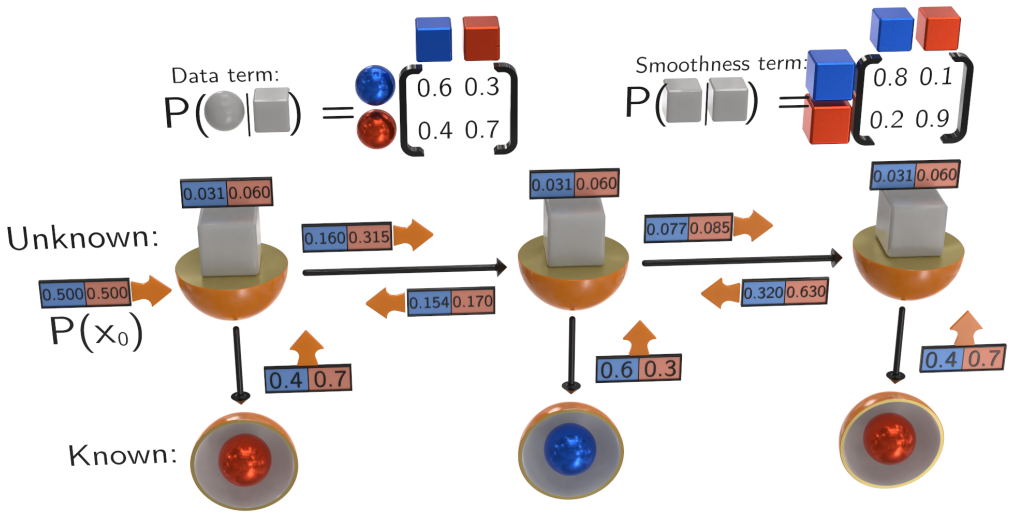

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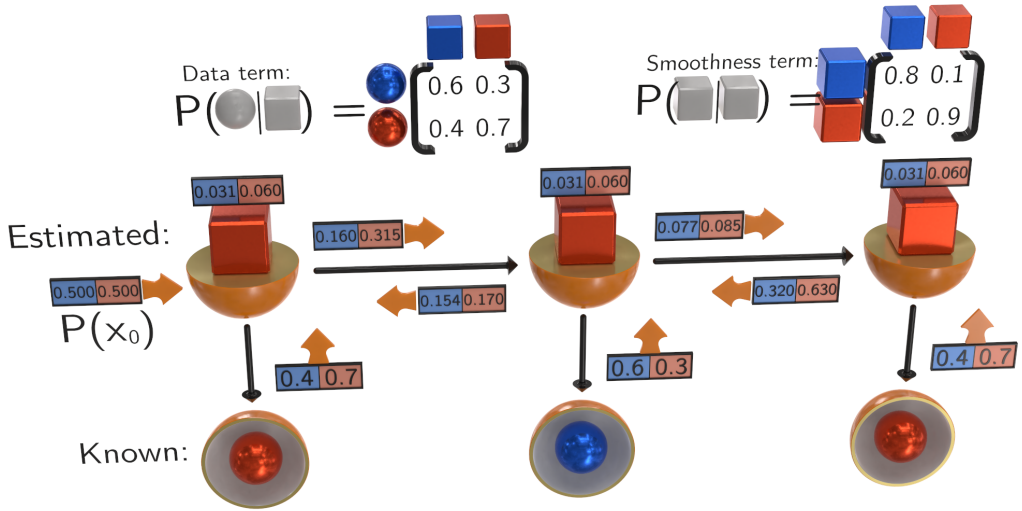

Maths

- Forwards pass:

$$M(x_{t+1})_{t\to t+1} = \max_{x_t} [P(x_{t+1}|x_t)P(y_t|x_t)M_{t-1\to t}(x_t)]$$

- Backwards pass:

$$M(x_{t-1})_{t\to t-1} = \max_{x_t} [P(x_t|x_{t-1})P(y_t|x_t)M_{t+1\to t}(x_t)]$$

- Belief: (output)

$$P(x_t) = M(x_t)_{t-1\to t}M(x_t)_{t+1\to t}P(y_t|x_t)$$

(Messages at chain ends: Replace with $P(x_0)$ or 1)

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Normalisation

- Numbers will underflow!
- One fix: Normalise messages

$$M(x_t)' = \frac{M(x_t)}{\sum_{x_t} M(x_t)}$$

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- Log space!
  (multiplication $\rightarrow$ addition $\therefore$ faster!)
- Preserves numerical stability... for a while (still breaks when chain gets too long)
- Normalise $\rightarrow$ subtract maximum from messages (= scale so max is 1)

$$\log M(x_t)' = \log M(x_t) - \max_{x_t} (\log M(x_t))$$

- Keep total $\log P(x)$ removed – can add back to get (log) probability of solution

Don't go backwards

- Last RV (x_{T-1}) correct after forward pass

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- Filtering = only need state of last node
  (as opposed to smoothing)
- Usually incremental
  – data/decisions in real time
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- Need maximum a posteriori? (MAP)
- Keep *pointers*:
  - Forward pass: Track winning x_{t-1} of each max
  - Backwards pass: Follow chain of pointers!
  - Faster (no computation)
- Handles ties
  (much better results when it matters)

n^{th} order

- Thus far: 1^{st} order
- 2^{nd} order: Depends on last two time steps

$$P(x_0, x_1, x_2, x_3, x_4) = P(x_4|x_3, x_2)P(x_3|x_2, x_1)P(x_2|x_1, x_0)P(x_1|x_0)P(x_0)$$

- And so on, 3^{rd} etc. (gets slow – rare)

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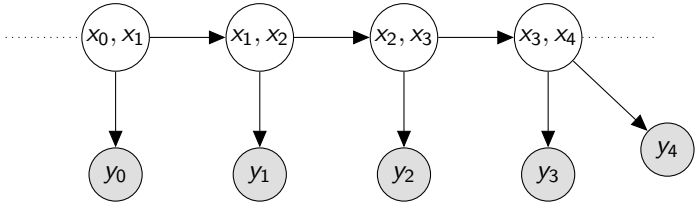
- And so on, 3^{rd} etc. (gets slow – rare)
- Convert to 1^{st} order:



$$P(x_{t+2}, x_{t+1}|x'_{t+1}, x_t) = P(x_{t+2}|x_{t+1}, x_t)\delta(x_{t+1} - x'_{t+1})$$

($\delta(.)$ = Kronecker delta function; can hang each y_t off either possible parent)

Notes

- Terminology:

$$-\log P(x) = \text{Energy} = \text{Cost}$$

- Note negative: max $\rightarrow$ min

- Wide enough window + ML can beat HMM
  - Has to learn Markov property
  - Needs lots of data

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Now

- Hidden Markov model
- Inferring HMM parameters
  (Baum–Welch)
- Kalman filter

Learning

- Need to learn:
  1. $P(x_0)$
  2. $P(y_t|x_t)$ (data term)
  3. $P(x_{t+1}|x_t)$ (smoothness term)

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- Three scenarios:
  1. Hidden state available to train: Trivial
  2. Never see hidden state: Baum–Welch
  3. Unaligned labels: Baum–Welch variant

Baum–Welch

- An Expectation-maximisation (EM) algorithm
  (see lecture 13, machine learning 1)

Baum–Welch

- An Expectation-maximisation (EM) algorithm
  (see lecture 13, machine learning 1)
- E-step: Calculate **marginals** using *forward–backwards*
  - Not Viterbi!
  - Another belief propagation specialisation
    (see lecture 12, machine learning 1)
  - MAP with max replaced by $\sum$
- M-step: Optimise *data* and *smoothness* terms to maximise probability
- Initialise; iterate until convergence

E-step I

- Just belief propagation/forward–backwards/dynamic programming:
  - Forwards pass:
$$M(x_{t+1})_{t\to t+1} = \sum_{x_t} [P(x_{t+1}|x_t)P(y_t|x_t)M_{t-1\to t}(x_t)]$$
  - Backwards pass:
$$M(x_{t-1})_{t\to t-1} = \sum_{x_t} [P(x_t|x_{t-1})P(y_t|x_t)M_{t+1\to t}(x_t)]$$

E-step II

- Assume $x_t \in \mathcal{S}$, $|\mathcal{S}| = N$, $\mathcal{T} = \{0, \ldots, T-1\}$
- Calculate:

$$\beta_t(s) = P(x_t = s) \propto M(x_t)_{t-1 \to t}[s] M(x_t)_{t+1 \to t}[s] P(y_t | x_t = s)$$

$$\rho_t(s, l) = P(x_t = s, x_{t+1} = l) \propto$$
$$M(x_t)_{t-1 \to t}[s] P(x_{t+1} = l | x_t = s) M(x_{t+1})_{t+2 \to t+1}[l] P(y_t | x_t = s) P(y_{t+1} | x_{t+1} = l)$$

  - $\beta_t(s) =$ belief
  - $\rho_t(s, l) =$ pairwise belief

(resolve $\propto$ by dividing through with $\sum$)

M-step

- x_t is discrete $\therefore$

$$A_{rc} = P(x_{t+1} = r | x_t = c)$$

- Assume y_t discrete $\therefore$

$$B_{rc} = P(y_t = r | x_t = c)$$

M-step

- x_t is discrete $\therefore$

$$A_{rc} = P(x_{t+1} = r | x_t = c)$$

- Assume y_t discrete $\therefore$

$$B_{rc} = P(y_t = r | x_t = c)$$

- M-step:

$$A_{rc} = \frac{\sum_{t \in \{0,\ldots,T-2\}} \rho_t(r,c)}{\sum_{t \in \mathcal{T}} \beta_t(c)}, \qquad B_{rc} = \frac{\sum_{\{t \in \mathcal{T}; y_t = r\}} \beta_t(c)}{\sum_{t \in \mathcal{T}} \beta_t(c)}$$

- Note: y_t can be continuous: Weighted density estimate

Initialisation

- Initialisation: Random PDFs
  (uniform will not work)
- Can get stuck in local minima
- Include prior

Initialisation

- Initialisation: Random PDFs
  (uniform will not work)
- Can get stuck in local minima
- Include prior

- Use log space
- Initial state:
  - One chain: Impossible
  - Many chains: Trivial

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Unaligned labels

- Know order of labels but not time
- Faster for human labeller

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Unaligned labels

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- Transform state space:
  - States = which training label
  - $P(x_{t+1}|x_t)$, smoothing term:
    Only allows you to stay on current label/move to next
- Fix first RV to be first label
- Fix last RV to be last label

- Baum–Welch as before
  (transform state space back to calculate pairwise beliefs)

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Kalman filter

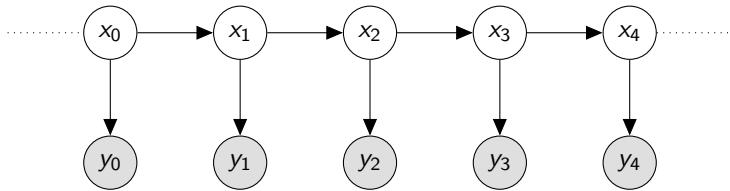
- Invented in 1960 by Kalman!
- Motivated by missiles (cold war)

- Used by Apollo
- Apollo development lead to *extended Kalman filter*

- These days:
  - Tracking with GPS + accelerometer...
    Cars, ships, planes, spacecraft, robots, *your phone*

Familiar?

- **Continuous** hidden Markov model
- **Gaussian** distributions
- Primarily for smoothing: Estimating true value from noisy measurements



- x: Unobserved (hidden) Markov chain
- y: Observations, dependent only on simultaneous hidden state

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Gaussian algebra

- Represent Gaussian as

$$\sigma[\mathbf{P}\mu, \mathbf{P}] \propto \exp\left[-\frac{1}{2}(x-\mu)^T\mathbf{P}(x-\mu)\right]$$

($\mathbf{P}\mu$ = p-mean, $\mathbf{P}$ = precision)

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- Offset:

$$\sigma[\mathbf{P}\mu, \mathbf{P}] + v = \sigma[\mathbf{P}\mu + \mathbf{P}v, \mathbf{P}]$$

- Scale:

$$\mathbf{S}\sigma[\mathbf{S}^{-\mathbf{T}}\mathbf{P}\mu, \mathbf{S}^{-\mathbf{T}}\mathbf{P}\mathbf{S}^{-1}]$$

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- Scale:

$$\mathbf{S}\sigma[\mathbf{S}^{-\mathbf{T}}\mathbf{P}\mu, \mathbf{S}^{-\mathbf{T}}\mathbf{P}\mathbf{S}^{-1}]$$

- Multiplication:

$$\sigma[\mathbf{P}_1\mu_1, \mathbf{P}_1]\sigma[\mathbf{P}_2\mu_2, \mathbf{P}_2] = \sigma[\mathbf{P}_1\mu_1+\mathbf{P}_2\mu_2, \mathbf{P}_1+\mathbf{P}_2]$$

- Addition:

$$\sigma[\mathbf{P}_1\mu_1, \mathbf{P}_1]+\sigma[\mathbf{P}_2\mu_2, \mathbf{P}_2] = \sigma[\mathbf{P}'(\mu_1+\mu_2), \mathbf{P}']$$

$$\mathbf{P}' = (\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1})^{-1}$$

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$$\sigma[\mathbf{P}\mu, \mathbf{P}] + v = \sigma[\mathbf{P}\mu + \mathbf{P}v, \mathbf{P}]$$

- Scale:

$$\mathbf{S}\sigma[\mathbf{S}^{-\mathbf{T}}\mathbf{P}\mu, \mathbf{S}^{-\mathbf{T}}\mathbf{P}\mathbf{S}^{-1}]$$

- Multiplication:

$$\sigma[\mathbf{P}_1\mu_1, \mathbf{P}_1]\sigma[\mathbf{P}_2\mu_2, \mathbf{P}_2] = \sigma[\mathbf{P}_1\mu_1+\mathbf{P}_2\mu_2, \mathbf{P}_1+\mathbf{P}_2]$$

- Addition:

$$\sigma[\mathbf{P}_1\mu_1, \mathbf{P}_1]+\sigma[\mathbf{P}_2\mu_2, \mathbf{P}_2] = \sigma[\mathbf{P}'(\mu_1+\mu_2), \mathbf{P}']$$

$$\mathbf{P}' = (\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1})^{-1}$$

- Extend with degenerate RV:

$$\text{Extend}_{1*}\left(\sigma[\mathbf{P}\mu, \mathbf{P}]\right) = \sigma\left[\begin{pmatrix}\mathbf{P}\mu \\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{pmatrix}\right]$$

- Marginalise: (sum over)

$$\text{Marg}_1\left(\sigma\left[\begin{pmatrix}\mathbf{h}_1 \\ \mathbf{h}_2\end{pmatrix}, \begin{pmatrix}\mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{12}^T & \mathbf{P}_{22}\end{pmatrix}\right]\right)$$

$$= \sigma[\mathbf{h}_2 - \mathbf{P}_{12}\mathbf{P}_{11}^{-1}\mathbf{h}_1, \mathbf{P}_{22} - \mathbf{P}_{12}\mathbf{P}_{11}^{-1}\mathbf{P}_{12}^T]$$

Transforms

- Data term:

$$y_t = H_t x_t + \sigma[\mathbf{0}, \mathbf{R}_t^{-1}]$$

  - H_t = mapping from true state to observed state
  - $\sigma[\mathbf{0}, \mathbf{R}_t^{-1}]$ = measurement noise

- "Smoothness" term:

$$x_{t+1} = F_t x_t + u_t + \sigma[\mathbf{0}, \mathbf{Q}_t^{-1}]$$

  - F_t = mapping from old state to new state
  - u_t = control signal (e.g. thrusters)
  - $\sigma[\mathbf{0}, \mathbf{Q}_t^{-1}]$ = noise

- Note: All linear

Message passing

- Gaussian algebra $\implies$ can just rearrange!
- All messages Gaussian

- Forwards:

$$M(x_{t+1})_{t\to t+1} = F_t\left(M(x_t)_{t-1\to t}H_t^{-1}\sigma[\mathbf{R}^{-1}y_t, \mathbf{R}^{-1}]\right) + u_t + \sigma[\mathbf{0}, \mathbf{Q}_t^{-1}]$$

- Backwards:

$$M(x_{t-1})_{t\to t-1} = F_t^{-1}\left[\left(M(x_t)_{t+1\to t}H_t^{-1}\sigma[\mathbf{R}^{-1}y_t, \mathbf{R}^{-1}]\right) - u_t + \sigma[\mathbf{0}, \mathbf{Q}_t^{-1}]\right]$$

- Belief:

$$P(x_t) = M(x_t)_{t-1\to t}M(x_t)_{t+1\to t}H_t^{-1}\sigma[\mathbf{R}^{-1}y_t, \mathbf{R}^{-1}]$$

Learning

- Problem specific

- Optimisation with data set (gradient descent)
- May involve calibration
  (experiments with real hardware)

SLAM

- "*Simultaneous localisation and mapping*"
  - Robots
  - Build/refines map of world
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- First that only used a camera:
  `https://www.youtube.com/watch?v=mimAWVm-0qA`
  (Kalman filter for camera location/rotation and feature locations)
- "*MonoSLAM: Real-Time Single Camera SLAM*",
  by Davison, Reid, Molton & Stasse (2007)

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From video

Notes

- Another demo: Load up Google maps and go for a walk!
- Not usually presented with Gaussian algebra

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  - F and H no longer linear – arbitrary functions
  - Updating mean as expected
  - Precision approximated: Updated using linear approximation around current point (First term of Taylor expansion, need Jacobian matrices)

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- Unscented Kalman filter:
  Better linear approximation using sampling

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Summary

- Hidden Markov models
- Baum–Welch to infer parameters
- Kalman filter/smoothing

Further reading

- Original Kalman filter paper:
  "*A New Approach to Linear Filtering and Prediction Problems*",
  by Kalman (1960) (worth reading, but find transcribed version!)

- If Gaussian is a bad assumption you need a particle filter:
  "*A Tutorial on Particle Filters for Online Nonlinear/Non-Gaussian Bayesian Tracking*",
  by Arulampalam, Maskell, Gordon & Clapp (2002)